Repeated Prophet Secretary

Mona Mohammadi **Raimundo Saona**

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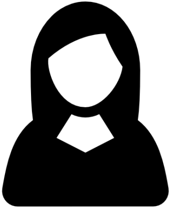
Job hunting

$v_1 \sim V_1$

$v_2 \sim V_2$



$v_3 \sim V_3$

Random order, distributions are known

Maximize the expected reward

Performance measure

For all instances,

$$\sup_{T \text{ stopping time}} \mathbb{E}[V_T] \geq \alpha \cdot \mathbb{E}\left[\max_{i \in [n]} V_i\right] .$$

Order	Performance	Reference
Random	[0.669, 0.732]	[CSZ21]
Adversarial	0.5	[KS77]
Free	[0.725, 0.745]	[PT22]

Distributions as prior information

Question

What can we do if we have less than the full distribution information?

Distributions $\rightarrow$ Samples

Assume that, instead of the distributions, we only have one sample from each distribution. Then,

Order	Samples	Performance	Reference
Random	1 per distribution	$1 - 1/e \approx 0.632$	[KNR19]
Adversarial	1 per distribution	0.5	[RWW20]
Free	??	??	

Revisiting the objective

- Samples come from history, i.e. past interactions.
- We optimize $\mathbb{E}[V_T]$, where T is a stopping time.
- This is a risk-averse measure.
- Also justified under repeated interactions.

Question

Should we not value the exploration of our stopping time?

Dynamic

Let us consider the following setting.

- The adversary chooses a sequence of values $v_1, v_2, \dots, v_{2n}$.
- Nature chooses a random permutation $\sigma \colon [2n] \to [2n]$.
- The player starts choosing in an online fashion:
  - During the first n elements, the player can choose one element.
  - During the second n elements, the player can choose one element.

The objective is to maximize the expectation of the sum of the picked values.

Example

Consider the values: $1, 2, 3, 4, 5, 10^6$.
A possible history of the game is as follows.

$$1$$

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$$1 \quad 5$$

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$$1 \quad 5 \quad \cancel{10^6}$$

Example

Consider the values: $1, 2, 3, 4, 5, 10^6$.
A possible history of the game is as follows.

$$1 \quad 5 \quad \sout{10^6}$$

$$3$$

Example

Consider the values: $1, 2, 3, 4, 5, 10^6$.
A possible history of the game is as follows.

$$\begin{matrix} 1 & 5 & \cancel{10^6} \\ 3 & 2 & \end{matrix}$$

Example

Consider the values: $1, 2, 3, 4, 5, 10^6$.

A possible history of the game is as follows.

$$\begin{array}{ccc} 1 & 5 & \cancel{10^6} \\ 3 & 2 & 4 \end{array}$$

Value obtained: $5 + 4$

Maximum value: $10^6 + 4$

Related literature

- We are no longer assuming that samples come from random distributions.
- [KNR19, CCES20] study this alternative model of randomness where part of the sequence is revealed as samples.
- In a single round, for k samples and n values, [CCES20] characterize the optimal limit performance

$$\lim_{\substack{k,n\to\infty \\ (k/n)\to p}} \alpha(k,n) = \alpha(p)\,.$$

Question

Having two rounds, how well can the gambler play?

UPPER bounds

Reduction to probability maximization

Consider an instance where $v_1 > v_2 > \ldots > v_n$. Moreover,

$$v_1 \gg v_2 \,.$$

Then, as $\lim v_2/v_1 = 0$, the performance α converges to the probability of picking the maximum value.

Upper bound for one round

Lemma

The optimal performance of prophet secretary is bounded by the maximum probability of picking the maximum value.

Corollary

For the prophet secretary with 0 *samples,*

$$\alpha \leq 1/e \approx 0.367 \,.$$

Corollary

For the prophet secretary with n samples,

$$\alpha \leq \ln(2) \approx 0.693 \,.$$

Upper bound for two rounds

Lemma

The optimal performance is bounded by the maximum probability of picking the maximum value, during both rounds.

Corollary

For the repeated prophet secretary with two rounds,

$$\alpha \leq \frac{1/e + \ln(2)}{2} \approx 0.530\,.$$

Upper bound, proof

Proof.

- Consider the instance with one extremely large value.
- During the first round, we can pick it with at most probability $1/e$ (Secretary problem).
- Moreover, we could collect at most n samples.
- During the second round, we can pick it with at most probability $\ln(2)$ (n-samples Secretary problem).
- Since the maximum value falls into each round with probability $1/2$, we conclude.

□

LOWER bounds

Good algorithm

Lemma

There exists an algorithm that obtains more than 0.5*, i.e. for the repeated prophet secretary with two rounds,*

$$\alpha \geq 0.500\,.$$

Good algorithm, construction

- During the first round, act secretary-like, observing a xn samples.
- During the second round, after observing $T_x n$ samples, apply the optimal algorithm that obtains $\alpha(p = T_x/(T_x + n))$.
- The final performance is of the form

$$\frac{-x \log(x) + \mathbb{E}\left[\alpha\left(\frac{T_x}{T_x+n}\right)\right]}{2},$$

where T_x is a random variable whose law depends on x.

- Maximize over x and conclude.

Questions

- Does the performance improve if we restrict the values to come from a single (unkown) distribution?
- Can we apply the same ideas to the repeated secretary problem?
- Is the proposed algorithm optimal?
- What happens with more rounds?

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